

Interferences with a Plane Parallel Plate Near the Critical Angle of Total Reflection

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The fringes of equal inclination with a plane parallel plate surrounded by an optically denser medium start at an angle of incidence less than the critical angle of total reflection. Despite its practical importance this effect was disregarded in optics up to now.

1. Introduction

Interference phenomena with a plane parallel plate were firstly observed by Herschel in 1809 and explained as fringes of equal inclination by Haidinger in 1849 [1]. These interferences become the physical basis of many optical techniques and devices such as, for instance, the Lummer-Gehrcke plate, the Fabry-Perot interferometer, interference filters, surface coatings and laser resonators. Despite its long history and its practical importance little attention was paid to the behaviour of the fringes of equal inclination near the critical angle of total reflection. This is intelligible since in most of the applications the case of nearly normal incidence is used, or the plate consists of an optically denser medium where no total reflection can be expected. However, this angular range is of great importance if optical parameters of thin layers were determined via critical angle measurements.

2. Theory

The reflection coefficient r (ratio of the amplitudes) of a plane parallel dielectric plate (medium 2) confined between two other media (medium 1 at the side of incidence and medium 3) is given by [2]

$$r = \frac{r_{12} + r_{23} e^{2i\beta}}{1 + r_{12} r_{23} e^{2i\beta}} \quad (1)$$

r_{lm} ($l, m = 1, 2, 3$) is the reflection coefficient of the interface between the media l and m and β is the

optical path difference between the two interfaces, where

$$\beta = k_{2z} d, \quad r_{lm} = \frac{k_l - k_m}{k_l + k_m} \quad (2)$$

with

$$k_l = \begin{cases} k_{lz}/\mu_l & \text{TE-case} \\ k_{lz}/\varepsilon_l & \text{TM-case} \end{cases}$$

ε_l and μ_l are the dielectric or magnetic permittivities, respectively, of the l -th medium. d is the thickness of the plane parallel plate. k_{lz} are the components of the wave vectors normal to the plate. They are connected by the refraction law which demands the invariance of the tangential component k_t of the wave vector when light of wave length λ passes through an interface between two media, when

$$k_{lz}^2 = \left(\frac{2\pi}{\lambda}\right)^2 \varepsilon_l - k_t^2, \quad k_t^2 = \left(\frac{2\pi}{\lambda}\right)^2 \varepsilon_1 \sin^2 \alpha. \quad (3)$$

α is the angle of incidence measured in the first medium. In the case of real k_{lz} , (1) shows an oscillating behaviour which describes the fringes of equal inclination mentioned above. For an optically less dense medium 2, there exists an angle of incidence α_c with $\sin^2 \alpha_c = \varepsilon_2/\varepsilon_1$, where k_{2z}^2 changes its sign, i.e. where k_{2z} becomes imaginary. Beyond this angle, r is less than unity because of the tunnel effect, and non-periodic, which means that there occur no interference fringes. Keep in mind that at a single boundary between two semi-infinite media total reflection would set in there.

Now we consider the transition region between oscillating and non-periodic behaviour in more detail. To begin with, we assume for simplicity that the media 1 and 3 are equal and that all media are non-magnetic. The reflection coefficient $R = |r|^2$ of

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the intensities is then for $k_z^2 \geq 0$ in a usual notation

$$R = \frac{4 R_{12} \sin^2 \beta}{(1 - R_{12})^2 + 4 R_{12} \sin^2 \beta} \quad (4)$$

with $R_{12} = r_{12}^2$.

It can easily be seen that R is minimum ($R = 0$) for $\beta = m\pi$ and maximum for $\beta = (m + \frac{1}{2})\pi$ if R is considered to be constant ($m = 0, 1, \dots$). For highly reflecting surfaces of the plate ($R \approx 1$) the reflection minima are very sharp. This leads to narrow dark fringes on an almost bright background in the reflected light and to narrow bright fringes on an almost dark background in the light transmitted through the plate. The positions of these fringes are characterized by an integer number $m = \beta/\pi$ called the order of interference. Using (2) and (3) we get the following expression for the order of interference:

$$m = \frac{2d}{\lambda} (\varepsilon_2 - \varepsilon_1 \sin^2 \alpha)^{1/2}. \quad (5)$$

Thus, for a given set of parameters ε_1 , ε_2 and λ the order of interference m can be changed by varying the thickness d of the plate or the angle of incidence α . Especially, the optical path difference β vanishes for zero thickness or for an angle of incidence equal to the critical angle α_c . In the latter case the accompanying change of the reflectivity R of the single interfaces according to (2) should be taken into account. Substituting $R_{12} = r_{12}^2$ by (2) we get the relation

$$R = \frac{(k_1^2 - k_2^2)^2 \sin^2 \beta}{4k_1^2 k_2^2 + (k_1^2 - k_2^2)^2 \sin^2 \beta}, \quad (6)$$

which expresses the angular dependence in a more lucid form than (4).

In order to determine the positions of the interference fringes we consider the zeros of the derivatives of R . As already mentioned, we have to distinguish between derivatives with respect to the thickness d and to the angle of incidence α . In the first case the reflectivity of the single surfaces is constant. Therefore the conditions for minimum and maximum reflectivity of the plate given above remain true. In the latter case we obtain

This leads to the condition $\sin \beta = 0$ for minimum reflectivity, i.e. for the dark fringes in the reflected light, and

$$\tan \beta = \beta \frac{k_1^2}{k_1^2 + k_2^2} \quad (8)$$

for maximum reflectivity between the dark fringes. Note that the denominator in (7) suppresses the zero of the derivative for $\beta = 0$. Indeed, for $\beta = 0$ the reflection coefficient R is neither maximum nor minimum. We obtain (TE case)

$$\lim_{\alpha \rightarrow \alpha_c} R = \frac{(k_1 d)^2}{4 + (k_1 d)^2}. \quad (9)$$

Hence, for a zero optical path difference ($\beta = 0$) the reflection coefficient of the plane parallel plate is zero only for $d = 0$ but not for $\alpha = \alpha_c$. This means that no interference fringe accompanies the critical angle, and, in dependence on the thickness, any value of R can be found there. The first visible fringe corresponds to the order of interference $m = 1$.

3. Results and Discussion

If the interference fringes are observed at a fixed angle and the interference pattern is scanned by a variation of the thickness, then the first dark fringe in the reflected light (or the first bright fringe in the transmitted light) corresponds to the order of interference $m = 0$. The first reflection maximum occurs for $m = \frac{1}{2}$. However, if the interference pattern for a constant thickness of the plate is scanned by the variation of the angle of incidence, then the first dark fringe in the reflected light (or the first bright fringe in the transmitted light) corresponds to the order of interference $m = 1$. The first reflection maximum appears between the interference fringes of the orders $m = 1$ and $m = 2$. Its angular position and the positions of all other maxima can be calculated by means of (8). From (8) it can be seen, too, that the reflection maxima do not appear in the middle between the reflection minima but are shifted towards the lower order of interference.

It is remarkable that the angular position which corresponds to the critical angle of total reflection

$$\frac{\partial R}{\partial \alpha} = - \frac{8 \sin \beta k_1 k_2 (k_1^2 - k_2^2) [k_1^2 \beta \cos \beta - (k_1^2 + k_2^2) \sin \beta]}{[(k_1^2 - k_2^2)^2 \sin^2 \beta + 4k_1^2 k_2^2]^2}. \quad (7)$$

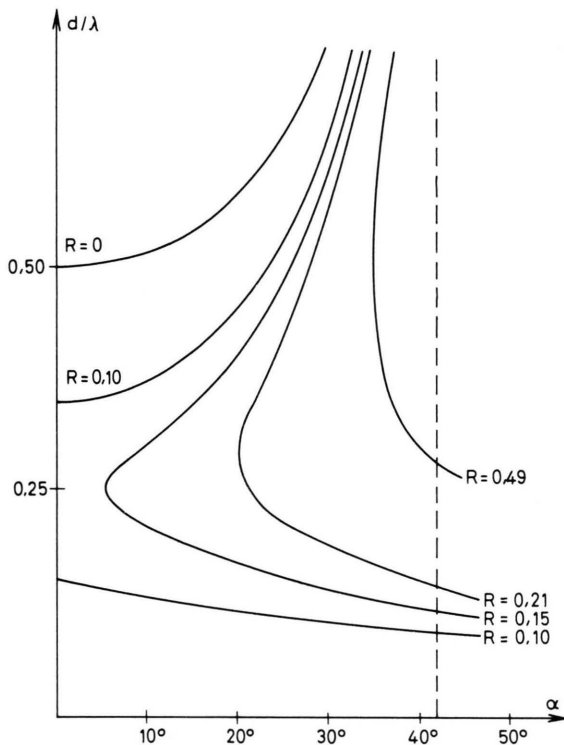


Fig. 1. Calculated dependence of the reflectivity R on the relative thickness d/λ and the angle of incidence α for an air plate between two glass prisms with $n = 1.5$. Solid lines: curves of equal reflectivity; dashed line: critical angle of total reflection α_c .

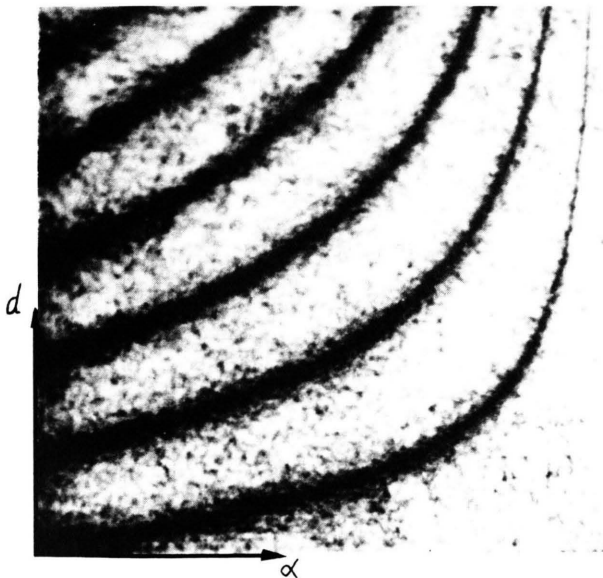


Fig. 2. Observed interference pattern of a slightly wedge shaped air plate between two glass prisms near the critical angle of total reflection.

for semi-infinite media is neither accompanied by an interference fringe nor by an other singularity nor by a constant value of the reflectivity and thus cannot be detected. For a layer not too thin the first interference fringe is in the immediate vicinity of the critical angle and very sharp. But if the rapid intensity decrease at the edge of this fringe is considered as critical angle of total reflection, then this would give rise to a systematic error in all parameters which are determined via critical angle measurements.

Moreover, according to (5) and (6) for decreasing thickness all interference fringes with $m \geq 1$ shift away from the critical angle, become broader and disappear in turn at an angular position which corresponds to normal incidence. The interference order $m = 0$ appear as darkening of the whole angular range in the reflected light or as brightening in the transmitted light. As fringe in the true sense this interference order can be observed only in an interference pattern of equal thickness, where it corresponds to the points of touch. These statements are illustrated by Fig. 1, which shows the calculated reflectivity R of a plane parallel air plate confined between two glass prisms in dependence on the thickness of the plate and the angle of incidence, respectively. The experimental evidence is presented in Fig. 2, which is a photograph of an interference pattern obtained with a slightly wedge shaped air plate between two glass prisms which was illuminated by a divergent laser beam.

Although formula (6) or analogous expressions can be found in many textbooks we believe that the behaviour of the fringes of equal inclination near the critical angle of total reflection discussed above was disregarded in optics up to now. We firstly observed this effect at a plane parallel liquid crystal layer [3, 4].

4. Conclusion

In many practically important cases the surfaces of the plane parallel plate are coated with metallic films or dielectric multilayers to enhance their reflectivity. This gives rise to sharper interference fringes necessary for a high resolving power, e.g. for a high finesse of a Fabry-Perot interferometer. In general (4), known as Airy formula, holds when phase jumps and absorption are taken into account [2]. But it can be apprehended, too, that no zeroth

order of interference can appear. The first interference fringe occurs at an angular position where the condition $\beta = \pi - \delta$ is satisfied. δ is the phase jump on internal reflection. Consequently, in this case, too, the critical angle of total reflection is not accompanied by an interference fringe and the first observable fringe is shifted away from the critical angle increasingly with decreasing thickness. Moreover, our discussion remains true for a slightly absorbing plate, i.e. if the characteristic absorption (or extinction) length is much greater than the thickness of the plate. This gives rise to a reduction of the number of interfering beams what leads only to a broadening of the interference fringes.

Finally we want to point out that the optical problem considered corresponds to the analogous quantum mechanical problem of the penetration of a particle through a rectangular potential barrier [5]. The sharp reflection minima and transmission maxima correspond to so-called virtual states, the first of which appears at an energy greater than the height of the potential barrier [6].

Above all it was this excellent agreement with quantum mechanics which encouraged us to reconsider the problem of interference with a plane parallel plate.

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